THE THEORY OF RESPONSE ANALYSIS OF FUZZY STOCHASTIC DYNAMICAL SYSTEMS WITH A SINGLE DEGREE OF FREEDOM

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SUMMARY

Most real-life structural/mechanical systems have complex geometrical and material properties and operate under complex fuzzy environmental conditions. These systems are certainly subjected to fuzzy random excitations induced by the environment. For an analytical treatment of such a system subjected to fuzzy random excitations, it becomes necessary to establish the general theory of dynamic response of a system to fuzzy random excitations. In this paper, the theory of response, fuzzy mean response and fuzzy covariance response of a single-degree-of-freedom (sdf) system to fuzzy random excitations in the time domain and frequency domain is put forward. The theory of response analysis of an sdf system to both stationary and non-stationary fuzzy random excitations in the time domain and frequency domain is established. Two examples are considered in order to demonstrate the rationality and validity of the theory, and the models of stationary filtered white noise and non-stationary filtered white noise fuzzy stochastic processes of the earthquake ground motion are set up. Methods of analysis for fuzzy random seismic response of sdf systems are put forward using the principles of response analysis of an sdf fuzzy random dynamic system.

KEY WORDS: fuzzy stochastic dynamic system; fuzzy random excitation; dynamic response; single-degree-of-freedom (sdf) system; fuzzy stochastic process

INTRODUCTION

A broad range of engineering problems involves analysis and design of systems subjected to excitations induced by the environment. The excitations may be caused by such diverse sources as the acoustic pressure field due to jet-noise, or boundary layer-noise; freestream turbulence of atmospheric and other flows; ocean waves; travel over a rough surface and earthquakes. A common feature of these sources of excitations is a large measure of uncertainty in their temporal and spatial characteristics. The uncertainty arises from both randomness and fuzziness, simultaneously. Let us consider two typical examples:

1. Consider a group of sampling points chosen at random to evaluate the degree of atmospheric pollution in a particular city. Some possible outcomes would be 'extreme pollution', 'serious pollution', 'light

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pollution', 'no pollution' and so on, and these results change with the time parameter t. In the problem randomness occurs because it is not known which outcome may be expected from any given sampling point. Once the outcome is available, there is still uncertainty about the precise meaning of the outcome. The latter uncertainty will be characterised by fuzziness. This is a typical fuzzy stochastic phenomenon with sustained time.

2. The earthquake loads specified for the design of aseismic structures depend mainly on the predicted earthquake intensity and the site soil classification. Earthquake excitation not only has evident randomness[1–5] but also has strong fuzziness[6–12] owing to the imprecision in the definition of earthquake intensity and site soil classification.

A realistic analysis and design of systems subjected to such excitations must account for the uncertainty arising from both randomness and fuzziness simultaneously in a consistent and rational manner.

The dynamical behaviour of a large class of systems subjected to fuzzy random excitations can be adequately predicted by discrete models having finite degrees of freedom. The mathematical equations describing the dynamic response of such a model consist of ordinary fuzzy random differential or integral equations.[9,11,13] If the governing equations are linear, the system is said to be a *discrete linear fuzzy random dynamical system*. In view of the analytical simplicity of ordinary fuzzy random differential equations, as compared to the partial fuzzy random differential equations that describe the behaviour of continuous fuzzy random dynamical systems, discrete models have received considerable attention.

Most real-life structural/mechanical systems have complex geometrical and material properties and operate under complex fuzzy environmental conditions. These systems are certainly subjected to fuzzy random excitations induced by the environment. For an analytical treatment of such a system subjected to fuzzy random excitations, it becomes necessary to construct its model, its environment, and the interaction between them. Discrete physical models of structural/mechanical systems subjected to fuzzy random excitations are usually constructed as an assemblage of idealized masses, springs and dashpots. For linear models, each of these elements is assumed to exhibit linear force deformation behaviour. The linear single-degree-of-freedom (sdf) fuzzy random dynamical system is the most important discrete model because (1) a large class of structural/mechanical systems subjected to fuzzy random excitations can be adequately modelled by it; and (2) the multiple-degree-of-freedom (mdf) and continuous models of systems subjected to fuzzy random excitations can be reduced to a set of sdf systems under fairly general conditions using the normal mode approach. In this paper, we shall discuss the behaviour of sdf fuzzy random dynamical systems in considerable detail using the results derived in References 11 and 13.

A BRIEF INTRODUCTION TO FUZZY STOCHASTIC PROCESSES

For convenience of the reader, in this section we state briefly some notions and results related to fuzzy random variables and fuzzy stochastic processes. These notions and results will be frequently referred to in the subsequent sections.

Let $\mathbf{R}$ be the real line and $(\mathbf{R}, \mathscr{B})$ the Borel measurable space.

Let $\mathscr{F}_0(\mathbf{R})$ denote the set of fuzzy subsets $\underset{\sim}{A}: \mathbf{R} \to [0, 1]$ with the following properties:

(1) $\{x \in \mathbf{R}: \underset{\sim}{A}(x) = 1\} \neq \emptyset$,

(2) $A_\alpha = \{x \in \mathbf{R}: \underset{\sim}{A}(x) \geqslant \alpha\}$ is a bounded closed interval[14] in $\mathbf{R}$ for each $\alpha \in (0, 1]$ i.e.,

$$A_\alpha = [A_\alpha^-, A_\alpha^+]$$

where $A_\alpha^- = \inf A_\alpha$, $A_\alpha^+ = \sup A_\alpha$, $A_\alpha^-, A_\alpha^+ \in A_\alpha$, $-\infty < A_\alpha^-$ and $A_\alpha^+ < +\infty$ for every $\alpha \in (0, 1]$. $A \in \mathscr{F}_0(\mathbf{R})$ is called a *bounded closed fuzzy number* (bcfn).

Definition 1.[15,16] Let $(\Omega, \mathscr{A}, \mathbf{P})$ be a probability space. A fuzzy set-valued mapping $\underset{\sim}{X}: \Omega \to \mathscr{F}_0(\mathbf{R})$ is called a *fuzzy random variable* (frv) if for every $B \in \mathscr{B}$ and every $\alpha \in (0, 1]$

$$X_\alpha^{-1}(B) = \{\omega \in \Omega;\ X_\alpha(\omega) \cap B \neq \emptyset\} \in \mathscr{A}.$$

A fuzzy set-valued mapping $\underset{\sim}{\mathbf{X}}:\Omega \to \mathscr{F}_0^m(\mathbf{R}) = \mathscr{F}_0(\mathbf{R}) \times \cdots \times \mathscr{F}_0(\mathbf{R})$, represented by $\underset{\sim}{\mathbf{X}}(\omega) = (\underset{\sim}{X}_1(\omega), \ldots, \underset{\sim}{X}_m(\omega))$, is called a *fuzzy random vector*[17,18] if for every k, $1 \leqslant k \leqslant m$, $\underset{\sim}{X}(k, \omega)$ is a fuzzy random variable.

A family of fuzzy random variables $\underset{\sim}{\mathbf{X}}(t) = \{\underset{\sim}{X}(t, \omega), t \in T\}$ is called a *fuzzy random function.*[15,17] Here the parameter set T could be any one of those: $\mathbf{R}$, $\mathbf{R}^+ = [0, \infty)$, $[a, b] \subset \mathbf{R}$, $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, $\{1, 2, \ldots, m\}$ and so on; in all these cases the parameter $t \in T$ may usually be thought of as time. If $T = \{1, 2, \ldots, m\}$, $\underset{\sim}{\mathbf{X}}(t)$ is a fuzzy random vector in the reference.[18] If $T = \mathbf{Z}$ or $\mathbf{Z}^+$, one sometimes speaks of a *fuzzy random sequence.* If $T = \mathbf{R}$ or $\mathbf{R}^+$ or $[a, b]$, $\underset{\sim}{\mathbf{X}}(t)$ is called a *fuzzy stochastic process.*[15,17] In all cases we give a general definition as follows.

Definition 2.[15] A fuzzy random function $\underset{\sim}{\mathbf{X}}(t) = \{\underset{\sim}{X}(t, \omega), t \in T\}$ is a fuzzy set-valued function from the space $T \times \Omega$ to $\mathscr{F}_0(\mathbf{R})$. $\underset{\sim}{X}(t, \cdot)$ is a frv on $(\Omega, \mathscr{A}, \mathbf{P})$ for every fixed $t \in T$; and $\underset{\sim}{X}(\cdot, \omega)$ is a fuzzy set-valued function with respect to the parameter set T for every fixed $\omega \in \Omega$, $\underset{\sim}{X}(\cdot, \omega)$ is called a *fuzzy sample function* or a *fuzzy trajectory.*

Definition 3.[11,13] (1) Let $(\Omega, \mathscr{A}, \mathbf{P})$ be a probability space and ζ a set-valued mapping

$$\zeta:\Omega \to I(\mathbf{R}) = \{[x, y]: x, y \in \mathbf{R}, x \leqslant y\}$$
$$\omega \mapsto \zeta(\omega) = [\zeta^-(\omega), \zeta^+(\omega)]$$

Then $\zeta(\omega) = [\zeta^-(\omega), \zeta^+(\omega)]$ is called a *random interval* if $\zeta^-(\omega)$ *and* $\xi^+(\omega)$ are both random variables on $(\Omega, \mathscr{A}, \mathbf{P})$.

(2) *Let* $T \subset \mathbf{R}$ and ζ be a set-valued mapping

$$\zeta: T \times \Omega \to I(\mathbf{R}) = \{[x, y]: x, y \in \mathbf{R}, x \leqslant y\}$$
$$(t, \omega) \mapsto \zeta(t, \omega) = [\zeta^-(t, \omega), \zeta^+(t, \omega)]$$

Then $\zeta(t, \omega) = [\zeta^-(t, \omega), \zeta^+(t, \omega)]$ is called a *random interval function* if $\zeta^-(t, \omega)$ and $\zeta^+(t, \omega)$ are both random functions i.e. when t is fixed $\zeta(t, \cdot) = [\zeta^-(t, \cdot), \zeta^+(t, \cdot)]$ is a random interval on $(\Omega, \mathscr{A}, \mathbf{P})$, when ω is fixed $\zeta(\cdot, \omega) = [\zeta^-(\cdot, \omega), \zeta^+(\cdot, \omega)]$ is an interval function[14] on T.

Theorem 1.[15] *Let* $\underset{\sim}{\mathbf{X}}(t) = \{\underset{\sim}{X}(t, \omega), t \in T\}$ *is a fuzzy random function if and only if for every* $\alpha \in (0, 1]$

$$\mathbf{X}_\alpha(t) = \{X_\alpha(t, \omega), t \in T\} = \{[X_\alpha^-(t, \omega), X_\alpha^+(t, \omega)], t \in T\}$$

is a random interval function for each $t \in T$ *and every* $\omega \in \Omega$, *and*

$$\underset{\sim}{X}(t, \omega) = \bigcup_{\alpha \in (0, 1]} \alpha X_\alpha(t, \omega) = \bigcup_{\alpha \in (0, 1]} \alpha [X_\alpha^-(t, \omega), X_\alpha^+(t, \omega)]$$

$$X_\alpha(t, \omega) = \bigcap_{n=1}^{\infty} X_{\alpha_n}(t, \omega) = \bigcap_{n=1}^{\infty} [X_{\alpha_n}^-(t, \omega), X_{\alpha_n}^+(t, \omega)]$$

where for every $\alpha \in (0, 1]$

$$X_\alpha^-(t, \omega) = \inf X_\alpha(t, \omega) = \inf\{x \in \mathbf{R}: \underset{\sim}{X}(t, \omega)(x) \geqslant \alpha\}$$
$$X_\alpha^+(t, \omega) = \sup X_\alpha(t, \omega) = \sup\{x \in \mathbf{R}: \underset{\sim}{X}(t, \omega)(x) \geqslant \alpha\}$$
$$\alpha_n = (1 - 1/(n + 1))\alpha$$

FUZZY RANDOM DYNAMICAL SYSTEM WITH A SINGLE DEGREE OF FREEDOM

Consider the sdf fuzzy random dynamical system shown in Figure 1. It consists of a mass, a massless spring, and a massless viscous damper. For linear behaviour, the spring force $F_s(t)$ is a linear function of the relative

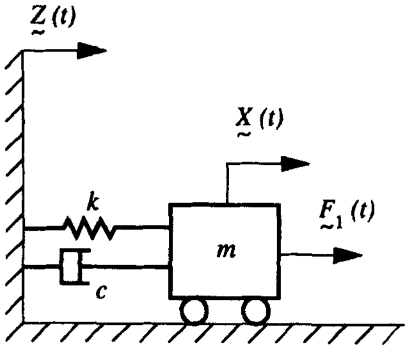

Figure 1. sdf fuzzy random dynamical system

fuzzy displacement $\underset{\sim}{Y}(t) = \underset{\sim}{X}(t) - \underset{\sim}{Z}(t)$:

$$F_s(t) = k\underset{\sim}{Y}(t) = k(\underset{\sim}{X}(t) - \underset{\sim}{Z}(t)) \tag{1}$$

and the damping force $F_D(t)$ is a linear function of relative fuzzy velocity $\dot{\underset{\sim}{Y}}(t) = \dot{\underset{\sim}{X}}(t) - \dot{\underset{\sim}{Z}}(t)$:

$$F_D(t) = c\dot{\underset{\sim}{Y}}(t) = c(\dot{\underset{\sim}{X}}(t) - \dot{\underset{\sim}{Z}}(t)) \tag{2}$$

where $\underset{\sim}{X}(t)$ is the absolute fuzzy displacement of the mass; $\underset{\sim}{Z}(t)$ is the fuzzy displacement of the base; k is the spring constant; and c is the viscous damping constant.

Let $\underset{\sim}{F}_1(t)$ be the fuzzy random load acting on the mass. The equations of motion of the system can be expressed in the following two ways:

$$m\ddot{\underset{\sim}{X}}(t) + c\dot{\underset{\sim}{X}} + k\underset{\sim}{X}(t) = c\dot{\underset{\sim}{Z}}(t) + k\underset{\sim}{Z}(t) + \underset{\sim}{F}_1(t) \tag{3}$$

or

$$m\ddot{\underset{\sim}{Y}}(t) + c\dot{\underset{\sim}{Y}}(t) + k\underset{\sim}{Y}(t) = -m\ddot{\underset{\sim}{Z}}(t) + \underset{\sim}{F}_1(t) \tag{4}$$

In both cases, the mathematical nature of the equations is identical. It is therefore perfectly general to consider the response of the following equation:

$$[Q_2(p)]\underset{\sim}{X}(t) = \ddot{\underset{\sim}{X}}(t) + 2\xi\omega_0\dot{\underset{\sim}{X}}(t) + \omega_0^2\underset{\sim}{X}(t) = \underset{\sim}{F}(t) \tag{5}$$

where

$$p = \mathrm{d}/\mathrm{d}t, \qquad \omega_0 = (k/m)^{1/2}, \qquad \xi = c/2m\omega_0$$

$$\underset{\sim}{X}(t) = \bigcup_{\alpha\in(0,1]} \alpha[X_\alpha^-(t), X_\alpha^+(t)] \tag{6}$$

$$\underset{\sim}{F}(t) = \bigcup_{\alpha\in(0,1]} \alpha[F_\alpha^-(t), F_\alpha^+(t)]$$

where $F_\alpha^{\mp}(t)$ have the units of acceleration and are given by

$$\underset{\sim}{F}(t) = 2\xi\omega_0\dot{\underset{\sim}{Z}}(t) + \omega_0^2\underset{\sim}{Z}(t) + \frac{\underset{\sim}{F}_1(t)}{m} \tag{7a}$$

or

$$\underset{\sim}{F}(t) = -\ddot{\underset{\sim}{Z}}(t) + \frac{1}{m}\underset{\sim}{F}_1(t) \tag{7b}$$

depending on whether $\underset{\sim}{X}(t)$ represents absolute or relative fuzzy displacement.

Equation (5) is an ordinary fuzzy random differential equation with constant coefficients of the same form as equation (4.3) in Reference 13. We discuss the time and frequency domain methods for determining the fuzzy response of such systems and derived the general expressions for the response fuzzy statistics of such systems in Reference 13.

From equations (6.15) in (6.16) in Reference 13 for every $\alpha \in (0, 1]$, equation (5) can be reduced to the following two conventional ordinary differential equations with constant coefficients:

$$\ddot{X}_\alpha^-(t) + 2\xi\omega_0 \dot{X}_\alpha^-(t) + \omega_0^2 X_\alpha^-(t) = F_\alpha^-(t) \tag{8}$$

$$\ddot{X}_\alpha^+(t) + 2\xi\omega_0 \dot{X}_\alpha^+(t) + \omega_0^2 X_\alpha^+(t) = F_\alpha^+(t) \tag{9}$$

Since equations (8) and (9) have constant coefficients, the independent integrals $X_{j,\alpha}^{\pm}(t), j = 1, 2$, are given by

$$X_{j,\alpha}^{\mp}(t) = \exp\{\lambda_{j,\alpha}(t - t_0)\}, \quad j = 1, 2$$

where the $\lambda_{j,\alpha}$ are the roots of the equation

$$\lambda^2 + 2\xi\omega_0\lambda + \omega_0^2 = 0 \tag{10}$$

The two roots of equation (10) are

$$\lambda_{1,2,\alpha} = -\xi\omega_0 \pm \mathrm{i}\omega_0\sqrt{1 - \xi^2} \tag{11}$$

The nature of the solution depends on the value ξ. For $\xi \geqslant 1$, the motion is called non-oscillatory; for $\xi > 1$, overdamped; for $\xi = 1$, critically damped; and for $\xi < 1$, oscillatory. In the last case, equation (11) can be expressed as

$$\lambda_{1,2,\alpha} = -\xi\omega_\mathrm{d} \pm \mathrm{i}\omega_\mathrm{d} \tag{12}$$

where $\omega_\mathrm{d} = \omega_0\sqrt{1 - \xi^2}$ is called the damped natural frequency.

The homogeneous parts of the solutions to equations (8) and (9) are given by

$$\begin{aligned} X_{h,\alpha}^{\pm}(t) &= C_{1,\alpha}^{\pm} X_{1,\alpha}^{\pm}(t) + C_{2,\alpha}^{\pm} X_{2,\alpha}^{\pm}(t) \\ &= C_{1,\alpha}^{\pm} \mathrm{e}^{(-\xi\omega_0 + \mathrm{i}\omega_\mathrm{d})(t - t_0)} + C_{2,\alpha}^{\pm} \mathrm{e}^{(-\xi\omega_0 - \mathrm{i}\omega_\mathrm{d})(t - t_0)} \\ &= \mathrm{e}^{-\xi\omega_0(t - t_0)}[C_{1,\alpha}^{\pm} \mathrm{e}^{\mathrm{i}\omega_\mathrm{d}(t - t_0)} + C_{2,\alpha}^{\pm} \mathrm{e}^{-\mathrm{i}\omega_\mathrm{d}(t - t_0)}] \end{aligned} \tag{13}$$

Let the initial conditions at $t = t_0$ be

$$X_{h,\alpha}^{\pm}(t_0) = X_{0,\alpha}^{\pm}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} X_{h,\alpha}^{\pm}(t)\bigg|_{t = t_0} = \dot{X}_{0,\alpha}^{\pm} \tag{14}$$

Substitute the initial conditions (14) in (13), solve for $C_{1,\alpha}$ and $C_{2,\alpha}$, and simplify:

$$X_{h,\alpha}^{\pm}(t) = g(t - t_0)X_{0,\alpha}^{\pm} + h(t - t_0)\dot{X}_{0,\alpha}^{\pm} \tag{15}$$

where

$$g(t) = \mathrm{e}^{-\xi\omega_0 t}\left[\cos\omega_\mathrm{d} t + \frac{\xi\omega_0}{\omega_\mathrm{d}}\sin\omega_\mathrm{d} t\right], \tag{16}$$

$$h(t) = \frac{1}{\omega_\mathrm{d}}\mathrm{e}^{-\xi\omega_0 t}\sin\omega_\mathrm{d} t \tag{17}$$

From equations (6.25) and (6.26) of Reference 13, $h_\alpha^\pm(t, \tau)$ are given by

$$h_\alpha^\pm(t,\tau) = \frac{\begin{vmatrix} e^{(-\xi\omega_0 + i\omega_d)\tau} & e^{(-\xi\omega_0 - i\omega_d)\tau} \\ e^{(-\xi\omega_0 + i\omega_d)t} & e^{(-\xi\omega_0 - i\omega_d)t} \end{vmatrix}}{\begin{vmatrix} e^{(-\xi\omega_0 + i\omega_d)\tau} & e^{(-\xi\omega_0 - i\omega_d)\tau} \\ (-\xi\omega_0 + i\omega_d)e^{(-\xi\omega_0 + i\omega_d)\tau} & (-\xi\omega_0 - i\omega_d)e^{(-\xi\omega_0 - i\omega_d)\tau} \end{vmatrix}} \tag{18}$$

which on simplification yields

$$h_\alpha^\pm(t,\tau) = h_\alpha^\pm(t-\tau) = \frac{1}{\omega_d} e^{-\xi\omega_0(t-\tau)} \sin \omega_d(t-\tau) \tag{19}$$

From equations (6.21) and (6.23) of Reference 13, the particular solutions of (8) and (9) are given by

$$X_{p,\alpha}^\pm(t) = \int_{t_0}^{t} h(t-\tau) F_\alpha^\pm(\tau)\,d\tau. \tag{20}$$

Hence the general solutions of equations (8) and (9) can be expressed as

$$X_\alpha^\pm(t) = g(t-t_0)X_{0,\alpha}^\pm + h(t-t_0)\dot{X}_{0,\alpha}^\pm + \int_{t_0}^{t} h(t-\tau)F_\alpha^\pm(\tau)\,d\tau \tag{21}$$

Substitution of equation (21) in (6) gives the fuzzy response of an sdf system to any given fuzzy random excitation $\underset{\sim}{F}(t)$:

$$\begin{aligned} \underset{\sim}{X}(t) &= \bigcup_{\alpha\in(0,1]} \alpha[X_\alpha^-(t), X_\alpha^\pm(t)] \\ &= \bigcup_{\alpha\in(0,1]} \alpha\Bigg[X_{0,\alpha}^- g(t-t_0) + \dot{X}_{0,\alpha}^- h(t-t_0) + \int_0^{t-t_0} h_\alpha^-(u)F_\alpha^-(t-u)\,du, \\ &\qquad X_{0,\alpha}^+ g(t-t_0) + \dot{X}_{0,\alpha}^+ h(t-t_0) + \int_0^{t-t_0} h_\alpha^+(u)F_\alpha^+(t-u)\,du \Bigg] \end{aligned} \tag{22}$$

where $g(t-t_0)$, $h(t-t_0)$ and $h^\pm(t-\tau)$ are given by equations (16), (17) and (19), respectively.

TIME DOMAIN METHOD

Let $\underset{\sim}{F}(t) = \bigcup_{\alpha\in(0,1]}\alpha[F_\alpha^-(t), F_\alpha^+(t)]$ be a fuzzy stochastic process satisfying the following two equations:

$$F_\alpha^\pm(t) = 2\xi\omega_0 \dot{Z}_\alpha^\pm(t) + \omega_0^2 Z_\alpha^\pm(t) + \frac{1}{m} F_{1,\alpha}^\pm(t) \tag{23}$$

$$F_\alpha^\pm(t) = -\ddot{Z}_\alpha^\pm(t) + \frac{1}{m} F_{1,\alpha}^\pm(t) \tag{24}$$

Further, let $\underset{\sim}{X}_0 = \bigcup_{\alpha\in(0,1]}[X_{0,\alpha}^-, X_{0,\alpha}^+]$ and $\dot{\underset{\sim}{X}}_0 = \bigcup_{\alpha\in(0,1]}\alpha[\dot{X}_{0,\alpha}^-, \dot{X}_{0,\alpha}^+]$ be fuzzy random variables.

1. *Fuzzy mean response*

Take the expectations of both sides of equation (21):

$$\begin{aligned} E(X_\alpha^\pm(t)) &= g(t-t_0)E(X_{0,\alpha}^\pm) + h(t-t_0)E[\dot{X}_{0,\alpha}^\pm] \\ &\quad + \int_{t_0}^{t} h(t-\tau)E(F_\alpha^\pm(\tau))\,d\tau \end{aligned} \tag{25}$$

Take the limit as $t_0 \to -\infty$; $g(t-t_0)$ and $h(t-t_0)$ tend to zero, and from equation (6.40) of Reference 13

$$\lim_{t_0 \to -\infty} E[X_\alpha^\pm(t)] = \int_0^\infty h(u) E[F_\alpha^\pm(t-u)]\,du \tag{26}$$

If $\underset{\sim}{F}(t)$ is stationary,[11,17] $E[F_\alpha^\pm(t)] = \mu_{F_\alpha^\pm}$ constants and equation (25) can be expressed as

$$E[X_\alpha^\pm(t)] = g(t-t_0)E[X_{0,\alpha}^\pm] + h(t-t_0)E[\dot{X}_{0,\alpha}^\pm] + \mu_{F_\alpha^\pm}\int_0^{t-t_0} h(u)\,du. \tag{27}$$

The integral in equation (27), with $h(t)$ given by equation (17), can be expressed as

$$\begin{aligned}
\int_0^{t-t_0} h(u)\,du &= \frac{1}{\omega_d}\int_0^{t-t_0} e^{-\xi\omega_0 u}\sin\omega_d u\,du \\
&= \frac{1}{\omega_d}\operatorname{Im}\left\{\int_0^{t-t_0} e^{(-\xi\omega_0 + i\omega_d)u}\,du\right\} \\
&= \frac{1}{\omega_d}\operatorname{Im}\left\{\frac{e^{(-\xi\omega_0+i\omega_d)(t-t_0)} - 1}{-\xi\omega_0 + i\omega_d}\right\} \\
&= \frac{1}{\omega_0^2}\left[1 - e^{\xi\omega_0(t-t_0)}\left\{\frac{\xi\omega_0}{\omega_d}\sin\omega_d(t-t_0) + \cos\omega_d(t-t_0)\right\}\right] \\
&= \frac{1}{\omega_0^2}[1 - g(t-t_0)].
\end{aligned}$$

Hence equation (27) can be expressed as

$$\begin{aligned}
E[X_\alpha^\pm(t)] = {}& g(t-t_0)E(X_{0,\alpha}^\pm) + h(t-t_0)E(\dot{X}_{0,\alpha}^\pm) \\
& + (\mu_{F_\alpha^\pm}/\omega_0^2)[1 - g(t-t_0)]
\end{aligned} \tag{28}$$

Take the limit as $t_0 \to -\infty$:

$$\lim_{t_0 \to -\infty} E(X_\alpha^\pm(t)) = \mu_{X_\alpha^\pm} = \mu_{F_\alpha^\pm}/\omega_0^2 \tag{29}$$

This result could be derived directly from equation (26) by setting $\mu_{F_\alpha^\pm}(t-u) = \mu_{F_\alpha^\pm}$ and carrying out the integrations.

Thus in all kinds of cases, the fuzzy mean response of an sdf system to any given fuzzy random excitations can be expressed in the time domain as follows:

$$E(\underset{\sim}{X}(t)) = \bigcup_{\alpha\in(0,1]} \alpha\{E(X_\alpha^-(t)), E(X_\alpha^+(t))\} \tag{30}$$

$$\lim_{t_0\to-\infty} E(\underset{\sim}{X}(t)) = \bigcup_{\alpha\in(0,1]} \alpha\left[\lim_{t_0\to-\infty} E(X_\alpha^-(t)), \lim_{t_0\to-\infty} E(X_\alpha^+(t))\right] \tag{31}$$

where $E(X_\alpha^\pm(t))$ and $\lim_{t_0\to-\infty} E(X_\alpha^\pm(t))$ are given by equation (25) or (28) and (26) or (29), respectively.

From equations (31) and (29), the fuzzy mean response of an sdf fuzzy stochastic dynamical system tends to an bcfn for a stationary fuzzy random excitation as $t_0 \to -\infty$. In particular, if the fuzzy mean of the fuzzy random excitation is zero, the fuzzy mean of the response also tends to zero after a sufficient length of time.

2. *Fuzzy covariance response*

Let the initial conditions $X_{0,\alpha}^{\pm}$, $\dot{X}_{0,\alpha}^{\pm}$ and $F_{\alpha}^{\pm}(t)$ be uncorrelated; then the covariance of the response is given by equation (6.46) of Reference 13 and can be expressed as

$$\begin{aligned} K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2) &= g(t_1 - t_0)g(t_2 - t_0)\sigma^2_{X_{0,\alpha}^\pm} + h(t_1 - t_0)h(t_2 - t_0)\sigma^2_{\dot{X}_{0,\alpha}^\pm} \\ &\quad + [g(t_1 - t_0)h(t_2 - t_0) + g(t_2 - t_0)h(t_1 - t_0)]K_{X_{0,\alpha}^\pm \dot{X}_{0,\alpha}^\pm} \\ &\quad + \int_{t_0}^{t_1}\int_{t_0}^{t_2} h(t_1 - \tau_1)h^*(t_2 - \tau_2)K_{F_\alpha^\pm F_\alpha^\pm}(\tau_1, \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \end{aligned} \tag{32}$$

where $\sigma_{X_{0,\alpha}^\pm}$ and $\sigma_{\dot{X}_{0,\alpha}^\pm}$ are the standard deviations of $X_{0,\alpha}^{\pm}$ and $\dot{X}_{0,\alpha}^{\pm}$, respectively. If $X_{0,\alpha}^{\pm}$ and $\dot{X}_{0,\alpha}^{\pm}$ are uncorrelated, $K_{X_{0,\alpha}^\pm \dot{X}_{0,\alpha}^\pm}$ in the third term in equation (32) vanishes. To simplify the algebra, we shall drop the first three terms in equation (32) in the subsequent derivations, so that

$$K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2) = \int_{t_0}^{t_1}\int_{t_0}^{t_2} h(t_1 - \tau_1)h^*(t_2 - \tau_2)K_{F_\alpha^\pm F_\alpha^\pm}(\tau_1, \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{33}$$

If $F_{\alpha}^{\pm}(t)$ is stationary, equation (33) reduces to

$$\begin{aligned} K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2) &= \int_{t_0}^{t_1}\int_{t_0}^{t_2} h(t_1 - \tau_1)h(t_2 - \tau_2)R_{F_\alpha^\pm F_\alpha^\pm}(\tau_2 - \tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &= \int_{0}^{t_1 - t_0}\int_{0}^{t_2 - t_0} h(u_1)h(u_2)R_{F_\alpha^\pm F_\alpha^\pm}(t_2 - t_1 + u_1 - u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 \end{aligned} \tag{34}$$

Take the limit as $t_0 \to -\infty$:

$$\begin{aligned} R_{X_\alpha^\pm X_\alpha^\pm}(t_2 - t_1) &= \lim_{t_0 \to -\infty} K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2) \\ &= \int_{0}^{\infty}\int_{0}^{\infty} h(u_1)h(u_2)R_{F_\alpha^\pm F_\alpha^\pm}(t_2 - t_1 + u_1 - u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 \end{aligned} \tag{35}$$

Hence in all kinds of cases, the fuzzy covariance response of an sdf system to any given fuzzy random excitation can be expressed in the time domain as follows:

$$K_{\underset{\sim}{X}\underset{\sim}{X}}(t_1, t_2) = \bigcup_{\alpha \in (0, 1]} \alpha\{K_{X_\alpha^- X_\alpha^-}(t_1, t_2), K_{X_\alpha^+ X_\alpha^+}(t_1, t_2)\} \tag{36}$$

where $K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2)$ are given by equation (33) or (34);

$$R_{\underset{\sim}{X}\underset{\sim}{X}}(t_2 - t_1) = \bigcup_{\alpha \in (0, 1]} \alpha\{R_{X_\alpha^- X_\alpha^-}(t_2 - t_1), R_{X_\alpha^+ X_\alpha^+}(t_2 - t_1)\} \tag{37}$$

where $R_{X_\alpha^\pm X_\alpha^\pm}(t_2 - t_1)$ are given by equation (35).

FREQUENCY DOMAIN METHOD

We shall now determine the fuzzy response statistics of an sdf system to fuzzy random excitation in the frequency domain using the results derived in Reference 11. From Example 6.4.1 in Reference 11, the particular solution of the sdf fuzzy random dynamical system equation (5) can be expressed as

$$\begin{aligned} \underset{\sim}{X}_p(t) &= \bigcup_{\alpha \in (0, 1]} \alpha[X_{p,\alpha}^-(t), X_{p,\alpha}^+(t)] \\ &= \bigcup_{\alpha \in (0, 1]} \alpha\left[\int_{-\infty}^{\infty} \bar{H}_\alpha^-(\omega, t)\,\mathrm{d}S_{F_\alpha^-}(\omega), \int_{-\infty}^{\infty} \bar{H}_\alpha^+(\omega, t)\,\mathrm{d}S_{F_\alpha^+}(\omega)\right] \end{aligned}$$

where

$$\bar{H}_\alpha^\pm(\omega, t) = H_\alpha^\pm(\omega)\mathrm{e}^{-\mathrm{i}\omega t} + H_\alpha^\pm(\omega)\mathrm{e}^{-\xi\omega_0(t-t_0)}\left[\frac{-\xi\omega_0 + \mathrm{i}(\omega-\omega_0)}{2\mathrm{i}\omega_\mathrm{d}}\mathrm{e}^{\mathrm{i}\omega_\mathrm{d}(t-t_0)} - \frac{-\xi\omega_0 + \mathrm{i}(\omega+\omega_\mathrm{d})}{2\mathrm{i}\omega_\mathrm{d}}\mathrm{e}^{-\mathrm{i}\omega_\mathrm{d}(t-t_0)}\right] \tag{38}$$

where

$$H_\alpha^\pm(\omega) = \frac{1}{(\omega_0^2 - \omega^2) - \mathrm{i}2\xi\omega_0\omega} = |H_\alpha^\pm(\omega)|\mathrm{e}^{\mathrm{i}\psi(\omega)} \tag{39}$$

$$|H_\alpha^\pm(\omega)|^2 = \frac{1}{(\omega_0^2 - \omega^2) + 4\xi^2\omega_0^2\omega^2}, \tag{40}$$

$$\psi(\omega) = \arctan\left(\frac{-2\xi\omega_0\omega}{\omega^2 - \omega_0^2}\right) \tag{41}$$

Figure 2 shows a plot of $|H_\alpha^\pm(\omega)|^2$ as a function of frequency ω for a small value of damping $\zeta \ll 1$. It is seen that $|H_\alpha^\pm(\omega)|^2$ is a sharply peaked function centred around the undamped natural frequency ω_0 at given threshold value level α. The magnitude of the function reduces to half the peak value (*half power height*), at a short distance $\xi\omega_0$ on either side of the frequency ω_0. The width $\Delta\omega = 2\xi\omega_0$ at the half power height is called the *system bandwidth*. It can be shown that the area under $|H_\alpha^\pm(\omega)|^2$ within the bandwidth $\Delta\omega$ is approximately $2/\pi \simeq 0{\cdot}636$ of the total area. It is clear from the above discussion that for $\xi \ll 1$, the sdf fuzzy random dynamical system defined by equation (5) acts as a narrowband fuzzy filter.

From equations (6.56)–(6.70) of Reference 13, the fuzzy covariance functions of the response, under different conditions, can be expressed as

$$K_{\underset{\sim}{X}\underset{\sim}{X}}(t_1, t_2) = \bigcup_{\alpha\in(0,1]} \alpha\{K_{X_\alpha^- X_\alpha^-}(t_1, t_2), K_{X_\alpha^+ X_\alpha^+}(t_1, t_2)\} \tag{42}$$

where

$$K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{H}_\alpha^\pm(\omega_1, t_1)\bar{H}_\alpha^{\pm *}(\omega_2, t_2)\,\hat{\Phi}_{F_\alpha^\pm F_\alpha^\pm}(\omega_1, \omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 \tag{43}$$

or

$$\begin{aligned} K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{H}_\alpha^\pm(\omega_1, t_1)\bar{H}_\alpha^{\pm *}(\omega_2, t_2)\Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega_1)\delta(\omega_2 - \omega_1)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 \\ &= \int_{-\infty}^{\infty} \bar{H}_\alpha^\pm(\omega_1, t_1)\bar{H}_\alpha^{\pm *}(\omega_1, t_2)\Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega_1)\,\mathrm{d}\omega_1 \end{aligned} \tag{44}$$

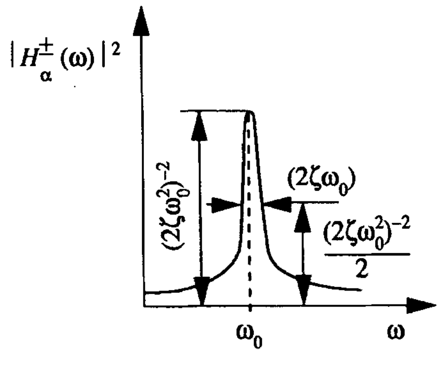

Figure 2. Frequency response function of a damped sdf fuzzy random dynamical system

and

$$R_{\underset{\sim}{X}\underset{\sim}{X}}(t_2 - t_1) = \bigcup_{\alpha \in (0,1]} \alpha\{R_{X_\alpha^- X_\alpha^-}(t_2 - t_1), R_{X_\alpha^+ X_\alpha^+}(t_2 - t_1)\} \tag{45}$$

where

$$R_{X_\alpha^\pm X_\alpha^\pm}(t_2 - t_1) = \lim_{t_0 \to -\infty} K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2) = \int_{-\infty}^{\infty} |H_\alpha^\pm(\omega)|^2 e^{i\omega(t_2 - t_1)} \Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega)\, d\omega \tag{46}$$

and

$$R_{\underset{\sim}{X}\underset{\sim}{X}}(0) = \bigcup_{\alpha \in (0,1]} \alpha[\sigma^2_{X_\alpha^-}, \sigma^2_{X_\alpha^+}] \tag{47}$$

where

$$R_{X_\alpha^\pm X_\alpha^\pm}(0) = \sigma^2_{X_\alpha^\pm} = \int_{-\infty}^{\infty} |H_\alpha^\pm(\omega)^2 \Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega)\, d\omega \tag{48}$$

or

$$R_{X_\alpha^\pm X_\alpha^\pm}(0) = \sigma^2_{X_\alpha^\pm} = \int_{-\infty}^{\infty} \Phi_{X_\alpha^\pm X_\alpha^\pm}(\omega)\, d\omega \tag{49}$$

From equation (6.61) of Reference 13,

$$\Phi_{X_\alpha^\pm X_\alpha^\pm}(\omega) = |H_\alpha^\pm(\omega)|^2 \Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega) \tag{50}$$

Equation (50) relating the power spectral density (psd) of the stationary response to the psd of the stationary excitation, at given threshold value level α, is of major significance in fuzzy random vibration theory.[9]

The nature of the psd of the response of a lightly damped sdf system to a broadband fuzzy random excitation at given threshold value level α is shown in Figure 3 for three typical cases. The following conclusions can be drawn from this figure:

1. An sdf fuzzy random dynamical system acts as a narrowband fuzzy filter centred at its natural frequency ω_0. The response of such a system is a narrowband random interval process $[X_\alpha^-(t), X_\alpha^+(t)]$ with ω_0 as the average frequency at given threshold value level α.
2. If the natural frequency ω_0 of an sdf fuzzy random system lies in the frequency range where input psd has a high value, the system response will be high, implying severe vibration. On the other hand, if the natural frequency ω_0 lies in a region where input psd is low, the system response will be small, implying weak vibration. These observations provide a basis for choosing the range in which the natural frequency ω_0 of an sdf fuzzy random system may lie, given the shape of the psd of the excitation. For example, in Figure 3 a frequency constraint $\omega_0 \geqslant \omega^{L}$ would ensure a low level of vibration response, and therefore an economical design.
3. If the natural frequency ω_0 of an sdf fuzzy random system lies in a frequency range where the psd of the excitation is a slowly varying function of frequency (cases 1 and 3 in Figure 3), the psd can be assumed to be locally constant (white) $\Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega_0)$, and equation (46) can be simplified to

$$R_{X_\alpha^\pm X_\alpha^\pm}(t_2 - t_1) \simeq \Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega_0) \int_{-\infty}^{\infty} |H_\alpha^\pm(\omega)|^2 e^{i\omega(t_2 - t_1)}\, d\omega \tag{51}$$

Clearly equation (51) is an acceptable approximation for a lightly damped system whose system bandwidth is small as compared to the region over which the excitation psd is flat. However, if ω_0 lies in a region where the excitation psd changes rapidly, as in case 2 of Figure 3, the approximation in equation (51) is not valid.

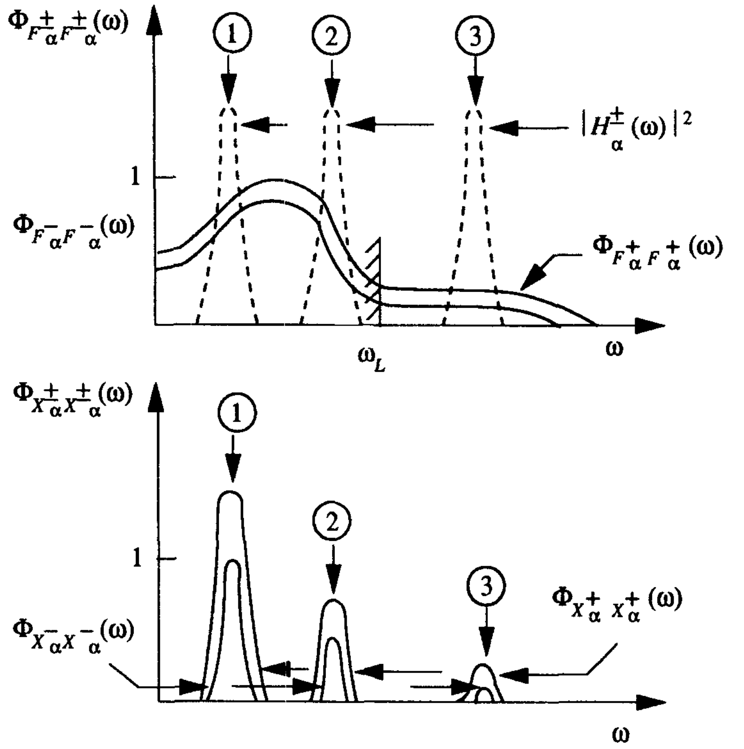

Figure 3. Graphical representation of the excitation psd $\Phi_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\omega)$ and the response psd $\Phi_{X^{\pm}_{\alpha}X^{\pm}_{\alpha}}(\omega)$ of an sdf fuzzy random dynamical system at given threshold value level α

It is clear from this that the frequency domain approach provided a useful insight into system behaviour, on the basis of which it is possible to simplify the analysis and design of sdf fuzzy random dynamical systems.

EXAMPLES

Example 1. Consider the sdf fuzzy random dynamical system of equation (5), and let $\underset{\sim}{F}(t)$ be white noise fuzzy stochastic process[11, 17] with

$$E(\underset{\sim}{F}(t)) = \underset{\sim}{0}, \qquad R_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\tau) = 2\pi\Phi_0^{\pm}\,\delta(\tau) \tag{52}$$

and let $\underset{\sim}{X}_0 = \dot{\underset{\sim}{X}}_0 = \underset{\sim}{0}$.

From equations (28), (30) and (52), it is clear that $E(\underset{\sim}{X}(t)) = \underset{\sim}{0}$. Substitute for $R_{F^{\pm}_{\alpha}F^{\pm}_{\alpha}}(\tau)$ from equation (52) in (34):

$$\begin{aligned} K_{X^{\pm}_{\alpha}X^{\pm}_{\alpha}}(t_1, t_2) &= 2\pi\Phi_0^{\pm}\int_0^{t_1-t_0}\int_0^{t_2-t_0} h(u_1)h(u_2)\,\delta(t_2 - t_1 + u_1 - u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 \\ &= 2\pi\Phi_\alpha^{\pm}\int_0^{t-t_0} h(u_1)h(|t_2 - t_1| + u_1)\,\mathrm{d}u_1 \end{aligned} \tag{53}$$

where $t = \min\{t_1, t_2\}$.

Take the limit as $t_0 \to -\infty$:

$$\begin{aligned} R_{X^{\pm}_{\alpha}X^{\pm}_{\alpha}}(t_2 - t_1) &= \lim_{t_0 \to -\infty} K_{X^{\pm}_{\alpha}X^{\pm}_{\alpha}}(t_1, t_2) \\ &= 2\pi\Phi_\alpha^{\pm}\int_0^{\infty} h(u)\,h(|t_1 - t_1| + u)\,\mathrm{d}u \end{aligned} \tag{54}$$

Substitute for $h(u)$ from equation (17) in (54):

$$R_{X_\alpha^\pm X_\alpha^\pm}(t_2 - t_1) = \frac{2\pi\Phi_0^\pm}{(\omega_d)^2} e^{-\xi\omega_0|t_2 - t_1|} \int_{-\infty}^{\infty} e^{-2\xi\omega_0 u} \sin\omega_d u \sin\omega_d(|t_2 - t_1| + u)\,du \tag{55}$$

By setting $t_2 - t_1 = \tau$ and evaluating the integral in equation (55), it can be shown that

$$R_{X_\alpha^\pm X_\alpha^\pm}(\tau) = \frac{\pi\Phi_0^\pm}{2\omega_0^3\xi} e^{-\xi\omega_0|\tau|}\left[\cos\omega_d|\tau| + \frac{\omega_0\xi}{\omega_d}\sin\omega_d|\tau|\right] \tag{56}$$

Set $\tau = 0$ in equation (56):

$$R_{X_\alpha^\pm X_\alpha^\pm}(0) = \sigma^2_{X_\alpha^\pm} = \pi\Phi_0^\pm/(2\omega_0^3\xi) \tag{57}$$

By setting $t_1 = t_2 = t$ in equation (54), the variance of the transient response can be expressed as

$$\sigma^2_{X_\alpha^\pm}(t) = 2\pi\Phi_0^\pm \int_0^{t-t_0} h^2(u)\,du \tag{58}$$

Substitute for $h(u)$ in equation (58) from (17):

$$\begin{aligned}\sigma^2_{X_\alpha^\pm}(t) &= \frac{2\pi\Phi_0^\pm}{\omega_d^2}\int_0^{t-t_0} e^{-2\xi\omega_0 u}[1 - \cos 2\omega_d u]\,du\\ &= \frac{\pi\Phi_0^\pm}{\omega_d^2}\int_0^{t-t_0} e^{-2\xi\omega_0 u}du - \frac{\pi\Phi_0^\pm}{\omega_d^2}\operatorname{Re}\int_0^{t-t_0} e^{-(2\xi\omega_0 - i2\omega_d)u}du\\ &= \frac{\pi\Phi_0^\pm}{2\xi\omega_0^3}\left\{1 - \frac{e^{-2\xi\omega_0(t-t_0)}}{\omega_d^2}[\omega_d^2 + 2(\xi\omega_0\sin\omega_d(t-t_0))^2\right.\\ &\qquad \left. + \xi\omega_0\omega_d\sin 2\omega_d(t-t_0)]\right\}\end{aligned} \tag{59}$$

Take the limit as $t_0 \to -\infty$; equation (59) reduces to (57), as it must.

Thus the fuzzy variance of the response can be expressed as

$$\begin{aligned}R_{\underset{\sim}{X}\underset{\sim}{X}}(\tau) = \bigcup_{\alpha\in(0,1]} \alpha\Bigg\{&\frac{\pi\Phi_0^-}{2\xi\omega_0^3}e^{-\xi\omega_0|\tau|}\left(\cos\omega_d|\tau| + \frac{\omega_0\xi}{\omega_d}\sin\omega_d|\tau|\right),\\ &\frac{\pi\Phi_0^+}{2\xi\omega_0^3}e^{-\xi\omega_0|\tau|}\left(\cos\omega_d|\tau| + \frac{\omega_0\xi}{\omega_d}\sin\omega_d|\tau|\right)\Bigg\}\end{aligned} \tag{60}$$

and

$$\sigma^2_{\underset{\sim}{X}} = \bigcup_{\alpha\in(0,1]} \alpha[\pi\Phi_0^-/(2\xi\omega_0^3), \pi\Phi_0^+/(2\xi\omega_0^3)] \tag{61}$$

If $\xi \ll 1$, terms involving products of ξ can be neglected in equation (59), and the following approximate expression can be derived:

$$\begin{aligned}\sigma^2_{\underset{\sim}{X}} &\simeq \bigcup_{\alpha\in(0,1]} \alpha\left[\frac{\pi\Phi_0^-}{2\xi\omega_0^3}(1 - e^{-2\xi\omega_0(t-t_0)}), \frac{\pi\Phi_0^+}{2\xi\omega_0^3}(1 - e^{-2\xi\omega_0(t-t_0)})\right]\\ &\simeq \bigcup_{\alpha\in(0,1]} \alpha\left[\frac{\pi\Phi_0^-}{\omega_0^2}(t - t_0), \frac{\pi\Phi_0^+}{\omega_0^2}(t - t_0)\right]\end{aligned} \tag{62}$$

if $2\xi\omega_0(t - t_0) \ll 1$.

Hence for $\xi \ll 1$, variances $\sigma^2_{X_\alpha^\pm}(t)$ increase almost linearly for small values of $t - t_0$. In all cases (except $\xi = 0$, for which variance is unbounded for $t_0 \to -\infty$), the variance is initially non-stationary and approaches the stationary value asymptotically as $t_0 \to -\infty$.

Example 2. We shall now consider the solution of Example 1 in the frequency domain. Since $\underset{\sim}{F}(t)$ is assumed to be white noise fuzzy stochastic process,

$$\Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega) = \Phi_0^\pm, \quad \forall \alpha \in (0, 1] \tag{63}$$

The covariance functions of the response are given by equations (44), (46) and (48) with $\Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega)$ replaced by $\Phi_0^\pm$. Thus

$$K_{X_\alpha^\pm X_\alpha^\pm}(t_1, t_2) = \Phi_0^\pm \int_{-\infty}^{\infty} \bar{H}_\alpha^\pm(\omega_1, t_1)\, \bar{H}_\alpha^{\pm *}(\omega_1, t_2)\, \mathrm{d}\omega_1 \tag{64}$$

where $\bar{H}_\alpha^\pm(\omega, t)$ are given by equation (38). From equations (48), (49) and (63),

$$R_{X_\alpha^\pm X_\alpha^\pm}(\tau) = \Phi_0^\pm \int_{-\infty}^{\infty} e^{i\omega\tau} \frac{\mathrm{d}\omega}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2} \tag{65a}$$

$$= \frac{\pi\Phi_0^\pm}{2\xi\omega_0^3} e^{-\xi\omega_0|\tau|}\left[\cos\omega_d|\tau| + \frac{\omega_0\xi}{\omega_d}\sin\omega_d|\tau|\right] \tag{65b}$$

Set $\tau = 0$ in equation (65a):

$$\sigma_{X_\alpha^\pm}^2 = R_{X_\alpha^\pm X_\alpha^\pm}(0) = \Phi_0^\pm \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2}$$

$$= \pi\Phi_0^\pm/(2\xi\omega_0^3) \tag{66}$$

Hence the fuzzy variance of the response is given by equations (60) and (61) under different conditions.

These results for white noise fuzzy random excitation hold also for a broadband fuzzy stochastic process with $\Phi_0^\pm$ replaced by $\Phi_{F_\alpha^\pm F_\alpha^\pm}(\omega)$ under conditions in which equation (51) is valid. This simplifies the analysis to a very large extent in the frequency domain method.

MODELS OF STATIONARY FILTERED WHITE NOISE FUZZY STOCHASTIC PROCESSES OF EARTHQUAKE GROUND MOTION[21]

Fourier analysis of existing strong-motion accelerograms reveal that the Fourier amplitude spectra are not constant with frequency even over a limited band. They are somewhat oscillatory in character, may peak at one or several frequencies, and damp out with increasing frequency—all of which suggests that a stationary filtered white noise fuzzy stochastic process $\underset{\sim}{a}(t) = \bigcup_{\alpha \in (0,1]} \alpha[a_\alpha^-(t), a_\alpha^+(t)]$ of limited duration could be more representative of actual strong ground motions provided the filter transfer characteristics are properly selected in Reference 21. Kanai and Tajimi have suggested the filter transfer function[1,4]

$$|H_{1,\alpha}^\pm(\omega)|^2 = \frac{1 + 4\xi_g^2(\omega/\omega_g)^2}{[1 - (\omega/\omega_g)^2]^2 + 4\xi_g^2(\omega/\omega_g)^2} \tag{67}$$

With this transfer function, the proper spectral density functions for the filtered processes $a_{1,\alpha}^-(t)$ and $a_{1,\alpha}^+(t)$ would be

$$\Phi_{a_{1,\alpha}^- a_{1,\alpha}^-}(\omega) = |H_{1,\alpha}^-(\omega)|^2 \Phi_{a_\alpha^-, a_\alpha^-}(\omega), \quad -\infty < \omega < \infty \tag{68}$$

$$\Phi_{a_{1,\alpha}^+ a_{1,\alpha}^+}(\omega) = |H_{1,\alpha}^+(\omega)|^2 \Phi_{a_\alpha^+ a_\alpha^+}(\omega), \quad -\infty < \omega < \infty \tag{69}$$

where $\Phi_{a_\alpha^- a_\alpha^-}(\omega)$ and $\Phi_{a_\alpha^+ a_\alpha^+}(\omega)$ are the power spectral density functions for $a_\alpha^{-1}(t)$ and $a_\alpha^+(t)$. Parameters ω_g and ξ_g appearing in equation (67) may be thought of as some characteristic ground frequency and characteristic damping ratio, respectively.

The first filtered fuzzy stochastic process of fuzzy stochastic process $\underset{\sim}{a}(t)$ can be expressed in the form

$$\underset{\sim}{a}_1(t) = \bigcup_{\alpha \in (0,1]} a[a_{1,\alpha}^-(t), a_{1,\alpha}^+(t)] \tag{70}$$

It should be recognized that the above filter attenuates the higher-frequency components and amplifies those frequency components in the neighborhood of ω_g. Since it does not change the amplitude as $\omega \to 0$, some difficulty may arise with the very low-frequency components. The cause of this difficulty can easily be recognized by noting that the power spectral density functions for ground velocity and ground displacement are obtained by dividing equations (68) and (69) by ω^2 and ω^4, respectively. Thus, strong singularities are present at $\omega = 0$ which cause the stationary variances of ground velocity and ground displacement to be unbounded. These undesirable singularities can be removed by passing processes $a_{1,\alpha}^{-}(t)$ and $a_{1,\alpha}^{+}(t)$ through another filter which greatly attenuates the very low-frequency components. An appropriate filter for two processes is one having the transfer function

$$|H_{2,\alpha}^{\pm}(\omega)|^2 = \frac{(\omega/\omega_1)^4}{[1-(\omega/\omega_1)^2]^2 + 4\xi_1^2(\omega/\omega_1)^2} \tag{71}$$

where the frequency parameter ω_1 and the damping parameter ξ_1 are selected to give the desired filter characteristics. The output processes $a_{2,\alpha}^{-}(t)$ and $a_{2,\alpha}^{+}(t)$ from this filter have power spectral density function of the form

$$\Phi_{a_{2,\alpha}^{-}a_{2,\alpha}^{-}}(\omega) = |H_2^{-}(\omega)|^2 \Phi_{a_2^{-}a_2^{-}}(\omega), \quad \omega \in (-\infty, \infty) \tag{72}$$

$$\Phi_{a_{2,\alpha}^{+}a_{2,\alpha}^{+}}(\omega) = |H_2^{+}(\omega)|^2 \Phi_{a_2^{+}a_2^{+}}(\omega), \quad \omega \in (-\infty, \infty) \tag{73}$$

where

$$|H_2^{\pm}(\omega)|^2 = |H_{1,\alpha}^{\pm}(\omega)|^2 |H_{2,\alpha}^{\pm}(\omega)|^2 \tag{74}$$

The second filtered fuzzy stochastic process of the fuzzy stochastic process $\underset{\sim}{a}(t)$ can be expressed in the form

$$\underset{\sim}{a}_2(t) = \bigcup_{\alpha\in(0,1]} a[a_{2,\alpha}^{-}(t), a_{2,\alpha}^{+}(t)] \tag{75}$$

The first filtering of processes $a_\alpha^{-}(t)$ and $a_\alpha^{+}(t)$ described above can be accomplished by solving the differential equations

$$\ddot{Y}_{r,\alpha}^{\pm} + 2\xi_g\omega_g \dot{Y}_{r,\alpha}^{\pm} + \omega_g^2 Y_{r,\alpha}^{\pm} = -a_{r,\alpha}^{\pm}(t), \quad r = 1, 2, \ldots \tag{76}$$

for $Y_{r,\alpha}^{\pm}$ and $\dot{Y}_{r,\alpha}^{\pm}$ using a digital computer and standard numerical-integration techniques and then obtaining $\underset{\sim}{a}_{1r}(t)$ using the relation

$$\underset{\sim}{a}_{1r}(t) = \bigcup_{\alpha\in(0,1]} \alpha[-2\xi_g\omega_g \dot{Y}_{r,2}^{-} - \omega_g^2 Y_{r,\alpha}^{-}, -2\xi_g\omega_g \dot{Y}_{r,\alpha}^{+} - \omega_g^2 Y_{r,\alpha}^{+}] \tag{77}$$

Likewise, the second filtering can be accomplished by solving the differential equations

$$\ddot{Z}_{r,\alpha}^{\pm} + 2\xi_1\omega_1 \dot{Z}_{r,\alpha}^{\pm} + \omega_1^2 Z_{r,\alpha}^{\pm} = -a_{1r,\alpha}^{\pm}(t), \quad r = 1, 2, \ldots \tag{78}$$

for $Z_{r,\alpha}^{\pm}$ and then letting

$$a_{2r}^{\pm}(t) = Z_{r,\alpha}^{\pm}$$

Therefore, the second filtered fuzzy stochastic process is given by

$$\underset{\sim}{a}_{2r}(t) = \bigcup_{\alpha\in(0,1]} \alpha[Z_{r,\alpha}^{-}, Z_{r,\alpha}^{+}] \tag{79}$$

All members of the desired stationary filtered white-noise fuzzy stochastic process can be obtained by repeating this procedure m times.

MODELS OF NON-STATIONARY FILTERED WHITE-NOISE FUZZY STOCHASTIC PROCESSES OF EARTHQUAKE GROUND MOTION[21]

To obtain an even more representative process for strong ground motions in Reference 21, the non-stationary character of actual accelerograms, the fuzziness of site soil classification and the fuzziness–randomness of the earthquake intensity can be considered, so that the ground acceleration is simulated as the product of a deterministic function for time t and a stationary filtered white noise fuzzy stochastic process, namely, a non-stationary fuzzy stochastic process $\underset{\sim}{a}(t)$ given by

$$\underset{\sim}{a}(t) = \sigma_\alpha(t)\underset{\sim}{a}_2(t) \tag{80}$$

where $\underset{\sim}{a}_2(t)$ is given by equation (75), and $\sigma_\alpha(t)$ is a deterministic function having an appropriate form based on statistical analysis of real accelerograms. One form[5] which has been suggested is that given by

$$\sigma_\alpha(t) = \sigma_{\max}\left[\frac{t}{c}\exp\left(1-\frac{t}{c}\right)\right]^b \tag{81}$$

where $\sigma_{\max} = \max(\sigma_\alpha(t))$ is the peak value of the function; c is the distance between the peak point and the origin of co-ordinates; b is a positive constant determining the form of the function curve.

FUZZY RANDOM SEISMIC RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM

In the previous two sections we have established the fuzzy stochastic process models of the earthquake ground motion. When the earthquake excitation is treated as a fuzzy stochastic process, the earthquake responses of structural systems are naturally also fuzzy stochastic processes. In this manner we are led to the study of fuzzy random vibrations of structural systems using the theory of fuzzy stochastic dynamical systems, which mainly deals with the characterization of input fuzzy random excitations to structural systems, determination of fuzzy random structural responses, and assessment of structural safety under such fuzzy random excitations.

Consider the structural system with a single degree of freedom (SDOF) subjected to earthquake generated ground acceleration $\ddot{\underset{\sim}{U}}_g(t) = \bigcup_{\alpha\in(0,1]}\alpha\{\ddot{U}^-_{g,\alpha}(t), \ddot{U}^+_{g,\alpha}(t)\}$, where $\ddot{U}^-_{g,\alpha}(t)$ and $\ddot{U}^+_{g,\alpha}(t)$ are two stationary white-noise processes of intensity Φ_0^- and Φ_0^+, respectively, the response of the linear SDOF structural system to this support acceleration $\ddot{\underset{\sim}{U}}_g(t)$ is governed by the equation:

$$\ddot{\underset{\sim}{U}}(t) + 2\xi\omega_0\dot{\underset{\sim}{U}}(t) + \omega_0^2\underset{\sim}{U}(t) = -\ddot{\underset{\sim}{U}}_g(t) \tag{82}$$

where $\underset{\sim}{U}(t)$ is the mass fuzzy displacement relative to the moving support. The principles for response analysis of a single-degree-of-freedom fuzzy random dynamic systems set forth in earlier sections give for an undercritically damped system

$$\begin{aligned} R_{\underset{\sim}{U}\underset{\sim}{U}}(\tau) = \bigcup_{\alpha\in(0,1]}\alpha\Bigg\{&\frac{\pi\Phi_0^-}{2\xi\omega_0^3}\mathrm{e}^{-\xi\omega_0|\tau|\cos\omega_d|\tau|} + \frac{\omega_0\xi}{\omega_d}\sin\omega_d|\tau|,\\ &\frac{\pi\Phi_0^+}{2\xi\omega_0^3}\mathrm{e}^{-\xi\omega_0|\tau|}\cos\omega_d|\tau| + \frac{\omega\xi}{\omega_\alpha}\sin\omega_d|\tau|\Bigg\} \end{aligned} \tag{83}$$

$$\begin{aligned} R_{\dot{\underset{\sim}{U}}\dot{\underset{\sim}{U}}}(\tau) = \bigcup_{\alpha\in(0,1]}\alpha\Bigg\{&\frac{\pi\Phi_0^-}{2\xi\omega_0^3}\mathrm{e}^{-\xi\omega_0|\tau|}\left(\cos\omega_d|\tau| - \frac{\xi}{\omega_d}\sin\omega_d|\tau|,\right.\\ &\left.\frac{\pi\Phi_0^+}{2\xi\omega_0}\mathrm{e}^{-\xi\omega_0|\tau|}\left(\cos\omega_d|\tau| - \frac{\xi}{\omega_d}\sin\omega_d|\tau|\right)\right)\Bigg\} \end{aligned} \tag{84}$$

$$\sigma^2_{\underset{\sim}{U}} = \bigcup_{\alpha\in(0,1]}\alpha\left[\frac{\pi\Phi_0^-}{2\xi\omega_0^3}, \frac{\pi\Phi_0^+}{2\xi\omega_0^3}\right] \tag{85}$$

$$\sigma^2_{\underset{\sim}{U}} = \bigcup_{\alpha\in(0,1]} \alpha\left[\frac{\pi\Phi_0^-}{2\xi\omega_0}, \frac{\pi\Phi_0^+}{2\xi\omega_0}\right] \tag{86}$$

$$\Phi_{U_\alpha^\pm U_\alpha^\pm}(\omega) = \frac{\Phi_0^\pm\,\omega_0^{-4}}{[1-(\omega/\omega_0)^2]^2 + 4\xi^2(\omega/\omega_0)^2} \tag{87}$$

$$\Phi_{\dot{U}_\alpha^\pm \dot{U}_\alpha^\pm}(\omega) = \frac{\Phi_0^\pm\,(\omega/\omega_0)^2\,\omega_0^{-2}}{[1-(\omega/\omega_0)^2]^2 + 4\xi^2(\omega/\omega_0)^2} \tag{88}$$

If a stationary filtered white-noise fuzzy stochastic process is assumed for ground acceleration $\ddot{\underset{\sim}{U}}(t) = \underset{\sim}{a}_2(t) = \bigcup_{\alpha\in(0,1]}\alpha[a_{2,\alpha}^-(t), a_{2,\alpha}^+(t)]$, and $a_{2,\alpha}^-(t)$ and $a_{2,\alpha}^+(t)$ are two stationary filtered white-noise processes having the power spectral densities $\Phi_{a_{2,\alpha}^- a_{2,\alpha}^-}(\omega)$ and $\Phi_{a_{2,\alpha}^+ a_{2,\alpha}^+}(\omega)$, respectively, equations (87) and (88) are still valid provided $\Phi_{a_{2,\alpha}^- a_{2,\alpha}^-}(\omega)$ and $\Phi_{a_{2,\alpha}^+ a_{2,\alpha}^+}(\omega)$ in equations (72) and (73) are substituted for Φ_0^- and Φ_0^+, respectively.

CONCLUSIONS

1. The theory of the response, fuzzy mean response and fuzzy covariance response of a single-degree-of-freedom (sdf) system to both stationary and non-stationary fuzzy random excitations in the time domain and frequency domain established in this paper has comprehensively taken account of the fuzziness and randomness of an sdf system.
2. The fuzziness and randomness of the earthquake intensity and the site soil classification must necessarily lead up to the fuzziness and randomness of the earthquake ground motion which acts as aseismic structural excitation. Therefore, it is proper that the models of stationary filtered white noise and non-stationary filtered white noise fuzzy stochastic processes of the earthquake ground motions are set up in the paper.
3. When the earthquake excitation is treated as a fuzzy stochastic process, the earthquake responses of structural systems are naturally also fuzzy stochastic processes. In this manner, we are led to the study of fuzzy random vibrations of structural systems using the theory of fuzzy stochastic dynamical systems, which mainly deals with the characterization of input fuzzy random excitations to structural systems, determination of fuzzy random structural responses, and assessment of structural safety under such fuzzy random excitations, and so the analysis methods for fuzzy random seismic response of single-degree-of-freedom systems are put forward using the principles for response analysis of a single-degree-of-freedom fuzzy random dynamic system given in the paper.

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